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# Homogeneous Law Invariant Coherent Multiperiod Value Measures and their Limits

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## 1 Introduction

Let  $(\Omega, \mathcal{F}, P)$  be a standard probability space. We denote  $L^p(\Omega, \mathcal{F}, P)$  by  $L^p$ ,  $1 \leq p \leq \infty$ .

**Definition 1** We say that a map  $\phi : L^\infty \rightarrow \mathbf{R}$  is a coherent value measure, if the following are satisfied.

- (1) If  $X \geq 0$ , then  $\phi(X) \geq 0$ .
- (2) Superadditivity :  $\phi(X_1 + X_2) \geq \phi(X_1) + \phi(X_2)$ .
- (3) Positive homogeneity : for  $\lambda > 0$  we have  $\phi(\lambda X) = \lambda\phi(X)$ .
- (4) For every constant  $c$  we have  $\phi(X + c) = \phi(X) + c$ .

Then Delbaen [6] essentially proved the following.

**Theorem 2** For  $\phi : L^\infty \rightarrow \mathbf{R}$ , the following conditions are equivalent.

- (1) There is a ( closed convex ) set of probability measures  $\mathcal{Q}$  such that any  $Q \in \mathcal{Q}$  is absolutely continuous with respect to  $P$  and for  $X \in L^\infty$

$$\phi(X) = \inf\{E^Q[X]; Q \in \mathcal{Q}\}.$$

- (2)  $\phi$  is a coherent value measure and satisfies the Fatou property, i.e., if  $\{X_n\}_{n=1}^\infty \subset L^\infty$  is uniformly bounded and converging to  $X$  in probability, then

$$\phi(X) \geq \limsup \phi(X_n).$$

- (3)  $\phi$  is a coherent value measure and satisfies the following property. If  $X_n$  is a uniformly bounded sequence that increases to  $X$ , then  $\phi(X_n)$  tends to  $\phi(X)$ .

Now we introduce the following notion.

**Definition 3** We say that a map  $\phi : L^\infty \rightarrow \mathbf{R}$  is law invariant, if  $\phi(X) = \phi(Y)$  whenever  $X, Y \in L^\infty$  have the same probability law.

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Let  $\mathcal{L}$  denote the set of probability measures on  $\mathbf{R}$ ,  $\mathcal{L}_p$ ,  $p \in [1, \infty)$ , denote the set of probability measures  $\nu$  on  $\mathbf{R}$  such that  $\int_{\mathbf{R}} |x|^p \nu(dx) < \infty$ , and  $\mathcal{L}_\infty$  denote the set of probability measures  $\nu$  on  $\mathbf{R}$  such that  $\nu(\mathbf{R} \setminus [-M, M]) = 0$  for some  $M > 0$ . Also,  $\mathcal{M}_{[0,1]}$  be the set of probability measure on  $[0, 1]$ .

For  $\nu \in \mathcal{L}$ , let  $F_\nu$  be the distribution functions of  $\nu$ , i.e.,  $F_\nu(z) = \nu((-\infty, z])$ ,  $z \in \mathbf{R}$ . Let us define  $Z : [0, 1] \times \mathcal{L} \rightarrow \mathbf{R}$  by

$$Z(x, \nu) = \inf\{z; F_\nu(z) > x\}, \quad x \in [0, 1], \nu \in \mathcal{L}.$$

Then  $Z(\cdot, \nu) : [0, 1] \rightarrow \mathbf{R}$  is non-decreasing and right continuous, and the probability law of  $Z(\cdot, \nu)$  under Lebesgue measure on  $[0, 1]$  is  $\nu$  (c.f.[9]). For any random variables  $X$ , we denote by  $\mu_X$  the probability law of  $X$ .

For each  $\alpha \in (0, 1]$ , let  $\eta_\alpha : \mathcal{L}_1 \rightarrow \mathbf{R}$  be given by

$$\eta_\alpha(\nu) = \alpha^{-1} \int_0^\alpha Z(x, \nu) dx, \quad \nu \in \mathcal{L}_1.$$

Also, we define  $\eta_0 : \mathcal{L}_\infty \rightarrow \mathbf{R}$  by

$$\eta_0(\nu) = \inf\{x \in \mathbf{R}; \nu((-\infty, x]) > 0\} \quad X \in \mathcal{L}_\infty.$$

Then we have the following (cf. [8], also see Section ).

**Theorem 4** Assume that  $(\Omega, \mathcal{F}, P)$  is a standard probability space and  $P$  is non-atomic. Let  $\phi : L^\infty \rightarrow \mathbf{R}$ . Then the following conditions are equivalent.

(1) There is a ( compact convex ) subset  $\mathcal{M}_0$  of  $\mathcal{M}_{[0,1]}$  such that

$$\phi(X) = \inf\left\{\int_0^1 \eta_\alpha(\mu_X) m(d\alpha); m \in \mathcal{M}_0\right\}, \quad X \in L^\infty.$$

(2)  $\phi$  is a law invariant coherent value measure with the Fatou property.

**Definition 5** We say that a map  $\eta : \mathcal{L}_\infty \rightarrow \mathbf{R}$  is a mild value measure (MVM), if there is a subset  $\mathcal{M}_0$  of  $\mathcal{M}_{[0,1]}$  such that

$$\eta(\nu) = \inf\left\{\int_0^1 \eta_\alpha(\nu) m(d\alpha); m \in \mathcal{M}_0\right\}, \quad \nu \in \mathcal{L}_\infty.$$

For any MVM  $\eta$ , we define a subset  $\mathcal{M}(\eta)$  of  $\mathcal{M}_{[0,1]}$  by

$$\mathcal{M}(\eta) = \{m \in \mathcal{M}; \eta(\nu) \leq \int_0^1 \eta_\alpha(\nu) m(d\alpha) \text{ for all } \nu \in \mathcal{L}_\infty\}.$$

For any  $\nu \in \mathcal{L}_1$ , we see that  $\eta_\alpha(\nu) \leq \eta_1(\nu)$ ,  $\alpha \in [0, 1]$ . So any MVM  $\eta$  can be extended to a map from  $\mathcal{L}_1$  to  $[-\infty, \infty)$  by

$$\eta(\nu) = \inf\left\{\int_0^1 \eta_\alpha(\nu) m(d\alpha); m \in \mathcal{M}(\eta)\right\}, \quad \nu \in \mathcal{L}_1.$$

We denote this map by the same symbol  $\eta$ .

**Definition 6** Let  $\eta$  be an MVM and  $(\Omega, \mathcal{F}, P)$  be a probability space.

(1) For any integrable random variable  $X$  and any sub- $\sigma$ -algebra  $\mathcal{G}$ , we define a  $\mathcal{G}$ -measurable random variable  $\eta(X|\mathcal{G})$  by

$$\eta(X|\mathcal{G}) = \eta(P(X \in dx|\mathcal{G})),$$

where  $P(X \in dx|\mathcal{G})$  is a regular conditional probability law of  $X$  given a sub- $\sigma$ -algebra  $\mathcal{G}$ . We call  $\eta(X|\mathcal{G})$  a conditional value measure.

(2) For any integrable random variable  $X$  and any filtration  $\{\mathcal{F}_k\}_{k=0}^n$ , we define an adapted process  $\{Z_k\}_{k=0}^n$  inductively by

$$Z_n = \eta(X|\mathcal{F}_n),$$

$$Z_{k-1} = \eta(Z_k|\mathcal{F}_{k-1}), \quad k = n, n-1, \dots, 1.$$

We denote an  $\mathcal{F}_0$ -measurable random variable  $Z_0$  by  $\eta(X|\{\mathcal{F}_k\}_{k=0}^n)$ , and call it a homogeneous filtered value measure.

(3) For any filtration  $\{\mathcal{F}_k\}_{k=0}^n$  and any integrable adapted process  $\{X_k\}_{k=0}^n$ , we define an adapted process  $\{Y_k\}_{k=0}^n$  inductively by

$$Y_n = X_n,$$

$$Y_{k-1} = X_{k-1} \wedge \eta(Y_k|\mathcal{F}_{k-1}), \quad k = n, n-1, \dots, 1.$$

We denote an  $\mathcal{F}_0$ -measurable random variable  $Y_0$  by  $\eta(\{X_k\}_{k=0}^n|\{\mathcal{F}_k\}_{k=0}^n)$ , and call it a homogeneous filtered value measure of an adapted process  $\{X_k\}_{k=0}^n$ .

In this paper, we consider two kinds of limit theorem for homogeneous filtered value measures. Let us introduce the following notion. For any MVM  $\eta$  and  $p \in [1, \infty)$ , let

$$\Delta_p(\eta) = \sup \left\{ \int_0^1 (\alpha^{-1/p} \wedge \frac{(1-\alpha)^{1-1/p}}{\alpha}) m(d\alpha); m \in \mathcal{M}(\eta) \right\}.$$

## 1.1 Brownian-Poisson Filtration

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space,  $\{B(t); t \in [0, \infty)\}$  be a  $d$ -dimensional Brownian motion and  $\{N_i(t); t \in [0, \infty)\}$ ,  $i = 1, \dots, \ell$ , be Poisson processes with an intensity  $\lambda_i$ . We assume that they are independent. Let  $\lambda = \sum_{i=1}^{\ell} \lambda_i$ , and let  $\mathcal{F}_t = \sigma\{B(s), N_i(s); s \leq t, i = 1, \dots, \ell\}$ ,  $t \geq 0$ .

Let  $\eta_n$ ,  $n = 1, 2, \dots$ , be MVM's. We assume the following.

(A-1) There is a constant  $C > 0$  such that  $\Delta_2(\eta_n) \leq C2^{-n/2}$ ,  $n = 1, 2, \dots$ .

Let  $F_0(y; \alpha, \beta)$ ,  $y \in \mathbf{R}^{\ell}$ ,  $0 \leq \alpha \leq \beta \leq 1$ , be given by

$$F_0(y; \alpha, \beta) = \inf \left\{ \int_0^{\gamma} Z(x, \lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i}) dx; \alpha \leq \gamma \leq \beta \right\}$$

$$= \inf \left\{ \gamma \eta_{\gamma}(\lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i}); \alpha \leq \gamma \leq \beta \right\},$$

and let  $b_n : \mathbf{R}^d \times \mathbf{R}^\ell \rightarrow \mathbf{R}$ ,  $n = 1, 2, \dots$ , be given by

$$b_n(x, y) = \inf \left\{ |x| 2^{n/2} \left( \int_0^1 \eta_\alpha(\mu_0) m(d\alpha) \right) \right. \\ \left. + \lambda \left( \int_0^1 m(d\alpha) \alpha^{-1} F_0(y; 0 \vee (1 - (2^n \lambda^{-1} (1 - \alpha))), 1 \wedge 2^n \lambda^{-1} \alpha) \right); m \in \mathcal{M}(\eta_n) \right\}.$$

Here  $\mu_0$  is a standard normal distribution.

Then  $b_n : \mathbf{R}^d \times \mathbf{R}^\ell \rightarrow \mathbf{R}$  is concave,

$$b_n(sx, sy) = s b_n(x, y), \quad x \in \mathbf{R}^d, \quad y \in \mathbf{R}^\ell, \quad s \geq 0,$$

and

$$b_n(x, y_1, \dots, y_\ell) \leq b_n(x', y'_1, \dots, y'_\ell),$$

if  $|x| \geq |x'|$ ,  $y_1 \leq y'_1, \dots, y_\ell \leq y'_\ell$ .

Let us assume the following furthermore..

(A-2) There is a continuous function  $b : \mathbf{R}^d \times \mathbf{R}^\ell \rightarrow \mathbf{R}$  such that  $b_n \rightarrow b$ ,  $n \rightarrow \infty$ , uniformly on compacts in  $\mathbf{R}^d \times \mathbf{R}^\ell$ .

Let  $K$  be a compact convex set in  $\mathbf{R}^d \times \mathbf{R}^\ell$  given by

$$K = \{(z, w) \in \mathbf{R}^d \times [0, \infty)^\ell; b(x, y) \leq x \cdot z + \sum_{i=1}^{\ell} \lambda_i y_i w_i \text{ for all } (x, y) \in \mathbf{R}^d \times \mathbf{R}^\ell\}.$$

Also, let  $\mathcal{K}$  be a set of martingales  $\rho(t)$  such that there are predictable processes  $\varphi : [0, \infty) \times \Omega \rightarrow \mathbf{R}^d$ ,  $\psi_i : [0, \infty) \times \Omega \rightarrow [0, \infty)$ ,  $i = 1, \dots, \ell$ , for which

$$P((\varphi(t), \psi_1(t), \dots, \psi_\ell(t)) \in K \text{ for any } t \in [0, T]) = 1$$

and

$$\rho(t) = \prod_{i=1}^{\ell} \left( \prod_{s \in (0, t], \Delta N_i(s) \neq 0} \psi_i(s) \right) \exp \left( \int_0^t \varphi(s) dB(s) - \frac{1}{2} \int_0^t |\phi(s)|^2 ds - \sum_{i=1}^{\ell} \lambda_i \int_0^t (\psi_i(s) - 1) ds \right),$$

$t \geq 0$ .

Then we have the following.

**Theorem 7** Under the assumption (A-1) and (A-2), we have the following.

For any  $X \in L^2(\Omega, \mathcal{F}_T, P)$ ,  $T > 0$ ,

$$\lim_{n \rightarrow \infty} \eta_n(X | \{\mathcal{F}_{2^{-n}k}\}_{k=0}^{2^{2n}}) = \inf \{E[\rho(T)X]; \rho \in \mathcal{K}\}.$$

We prove this theorem in Section 5 via a nonlinear partial differential equation.

## 1.2 Collective Risk

Let  $(\Omega, \mathcal{F}, P)$  be a probability space. Let  $K \geq 1$ ,  $p \in (1, \infty)$ ,  $p_k \in \mathbf{R}$ ,  $\lambda_k > 0$ , and  $\nu_k \in \mathcal{L}_p$ ,  $k = 1, \dots, K$ . Let  $Z_i^{(k)}$ ,  $\tau_i^{(k)}$ ,  $k = 1, \dots, K$ ,  $i = 1, 2, \dots$ , be independent random variables such that the distribution of  $Z_i^{(k)}$  is  $\nu_k$ , and  $P(\tau_i^{(k)} > t) = \exp(-\lambda_k t)$ ,  $t \geq 0$ , for  $k = 1, \dots, K$ ,  $i = 1, 2, \dots$ . Let  $N_i^{(k)}(t) = 1_{\{\tau_i^{(k)} \leq t\}}$ , and  $X_i^{(k)}(t) = Z_i^{(k)} N_i^{(k)}(t) + p_k(\tau_i^{(k)} \wedge t)$  for  $t \geq 0$ ,  $k = 1, \dots, K$ ,  $i = 1, 2, \dots$ .

Let  $\mathcal{F}_t = \sigma\{X_i^{(k)}(s); s \in [0, t], k = 1, \dots, K, i = 1, 2, \dots\}$ ,  $t \geq 0$ . Also, let

$$X(t; m_1, \dots, m_K) = \sum_{k=1}^K \sum_{i=1}^{m_k} X_i^{(k)}(t)$$

for any  $t \geq 0$ , and any  $m_1, \dots, m_K \in \mathbf{Z}_{\geq 0}$ . Here  $\mathbf{Z}_{\geq 0}$  denotes the set of non-negative integers.

**Theorem 8** Let  $\eta$  be MVM. Assume that  $\Delta_p(\eta) < \infty$ . Let  $\Phi : [0, \infty)^K \times \mathbf{R}^K \rightarrow \mathbf{R}$  be given by

$$\Phi(x, \xi) = \eta\left(\sum_{\ell_1, \dots, \ell_K=0}^{\infty} \left(\prod_{k=1}^K (\exp(-\lambda_k x_k) \frac{(\lambda_k x_k)^{\ell_k}}{\ell_k!})\right) (\nu_1 - \xi_1)^{\ast \ell_1} \cdots (\nu_K - \xi_K)^{\ast \ell_K}\right) + \sum_{k=1}^K p_k x_k,$$

for  $x \in [0, \infty)^K$ ,  $\xi \in \mathbf{R}^K$ . Here  $\ast$  stands for the convolution and  $\nu + a$  denotes a probability measure on  $\mathbf{R}$  given by the following for any probability measure  $\nu$  on  $\mathbf{R}$  and  $a \in \mathbf{R}$ .

$$(\nu + a)(A) = \nu(\{x \in \mathbf{R}; x - a \in A\}) \text{ for any Borel set } A \text{ in } \mathbf{R}.$$

Assume that there is a  $C^1$  function  $u : [0, \infty) \times [0, \infty)^K \rightarrow \mathbf{R}$  such that  $u(0, x) = 0$ ,  $x \in [0, \infty)^K$ , and satisfies the following Hamilton-Jacobi equation

$$\frac{\partial}{\partial t} u(t, x) = \Phi(x, \frac{\partial}{\partial x^1} u(t, x), \dots, \frac{\partial}{\partial x^K} u(t, x)), \quad (t, x) \in [0, \infty) \times [0, \infty)^K.$$

Then we have the following.

$$\sup\{|h\eta(X(t; m_1, \dots, m_K) | \{\mathcal{F}_{jh}\}^{[h-2]}) - u(t, m_1 h, \dots, m_K h)|;$$

$$t, m_1 h, \dots, m_K h \in [0, R], m_1, \dots, m_K \in \mathbf{Z}_{\geq 0}\} \rightarrow 0,$$

as  $h \downarrow 0$ , for any  $R > 0$ .

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